

This shows us two things simultaneously

1) fermion masses can be "generated" by a spontaneous symmetry breaking  
Higgs Mechanism if continuous symmetry is gauged.

2) for a global symmetry, we see it naturally gives rise to a coupling of a fermion (nucleon) to a pseudo-scalar ( $\pi$ )

from here  $\rightarrow$  (125)

Field redefinition Thursday 20 Feb

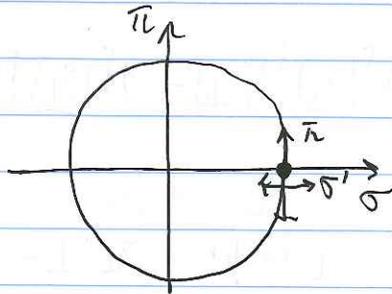
I have eluded to different choices of parameterizing the complex scalar theory. One of the most important points: the physics must be independent of the parameterization:

S-matrix elements are physical observables

$$S = 1 + iT$$

We discussed complex scalar in Hermitian basis

$$\phi = \frac{\sigma + i\pi}{\sqrt{2}}, \quad \phi^\dagger = \frac{\sigma - i\pi}{\sqrt{2}}$$



Spontaneous symmetry  
breaking  $\mu^2 < 0$

$$\sigma^2 + \pi^2 = v^2$$

There is perhaps a more natural parameterization

$$\phi = \frac{v + \rho}{\sqrt{2}} e^{i\theta/v} = \frac{v + \rho(x)}{\sqrt{2}} e^{i\theta(x)/v}$$

This is a non-linear realization of the theory. The  $\mathcal{L}$  interactions may look quite different, but the resulting S-matrix elements must be the same.

Under a  $U(1)$  transformation

$$\rho \rightarrow \rho$$

$$\theta \rightarrow \theta + \beta v$$

The scalar  $\mathcal{L}$

$$\mathcal{L}_\phi = \frac{1}{2} (\partial_\mu \rho)^2 + \frac{1}{2} (\partial_\mu \theta)^2 \left[ 1 + \frac{\rho}{v} \right]^2 - \left[ \frac{-\mu^4}{4\lambda} - \mu^2 \rho^2 + \lambda v \rho^3 + \frac{\lambda}{4} \rho^4 \right]$$

Notice the similarities and differences

$\rho, \sigma$  massive  
 $\theta, \pi$  massless

$\mathcal{L}_{p.o} \Rightarrow$  derivative interactions

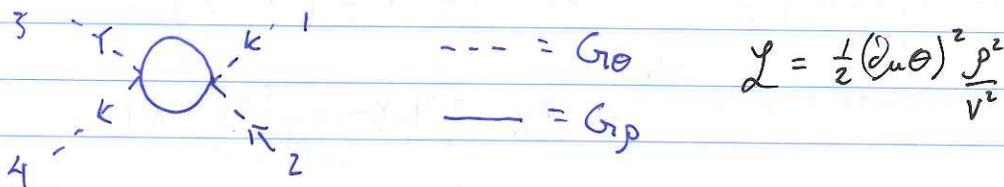
$$\frac{1}{2}(\partial_\mu \theta)^2 \left[ \frac{p^\mu p^\mu}{v^2} + 1 \right]^2 = \frac{1}{2}(\partial_\mu \theta)^2 \left[ 1 + 2\frac{p^\mu p^\mu}{v^2} + \frac{p^2}{v^2} \right]$$

This Lagrangian is "not renormalizable"

There are dimension 5 & 6 operators.

If we calculate QM corrections, we will need to introduce new operators in the  $\mathcal{L}$  to absorb the divergences.

Let us examine one higher order correction to see an explicit example.



s-channel contribution to  $\theta\theta$  scattering

$$iA(s) = \frac{1}{2} \times 2 \times \left( \frac{-i}{2v^2} \right)^2 \times N_W (-ip_1) \cdot (-ip_2) (ip_3) (ip_4) \cdot \int \frac{d^4 k}{(2\pi)^4} \frac{i}{(p+k)^2 - m^2 + i\epsilon} \frac{i}{k^2 - m^2 + i\epsilon}$$

2nd order perturbation theory  
 ordering of  $\mathcal{L}$  insertions

$$\partial_\mu \theta = \partial_\mu \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2p_0} \left[ a_p^\dagger e^{-ip \cdot x} + a_p e^{+ip \cdot x} \right]$$

$$\langle 00 | \mathcal{L}_x \mathcal{L}_y | 00 \rangle$$

or

$$\langle 00 | \mathcal{L}_y \mathcal{L}_x | 00 \rangle$$

we want annihilation operator  
 $\partial_\mu \theta | 0 \rangle = \partial_\mu \theta \theta^\dagger | 0 \rangle$

↑ This uses creation operator so non-vanishing contribution from annihilation from  $\partial_\mu \theta$

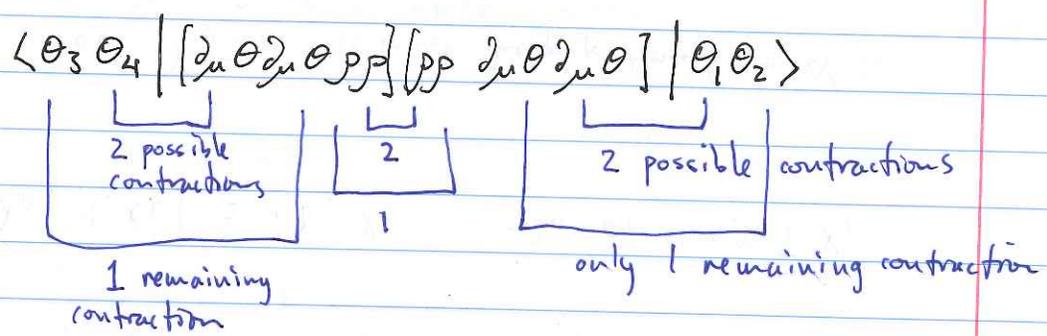
$$\partial_\mu \theta = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2p_0} \cdot \left[ -i p_\mu a_p e^{-ip \cdot x} + i p_\mu a_p^\dagger e^{+ip \cdot x} \right]$$

for out going states, it is the creation term we want

$$\langle 0 | \partial_\mu \theta = \langle 0 | \theta \partial_\mu \theta$$

$\uparrow$                        $\uparrow$   
 $a_p$                       want  $a_p^\dagger \Rightarrow +i p_\mu$

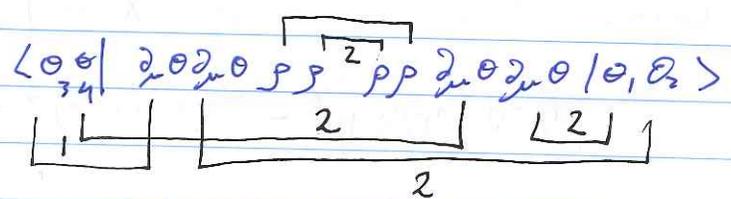
How about  $N_w = ?$  This is a factor from the different Wick contractions



$$\Rightarrow 8 (-i p_1) \cdot (-i p_2) (+i p_3) \cdot (+i p_4) \quad P_1 \cdot P_2 = \frac{1}{2} [(P_1 + P_2)^2 - P_1^2 - P_2^2]$$

$$= \frac{1}{2} S - \phi$$

~~What about other contractions?~~ What about other contractions?



$$\Rightarrow (P_1 \cdot P_3)(P_2 \cdot P_4) \quad \frac{1}{i} \quad (P_1 \cdot P_4)(P_2 \cdot P_3)$$

t-channel                      u-channel



For now, focus on S-channel

$$iA(s) = (-)^4 (i)^8 \cdot 8 \frac{P \cdot P_2 P_3 \cdot P_4}{(S/2)^2} \int_R d^4 k \int_0^1 dx \frac{1}{[(k^2 - m^2 + i\epsilon)(1-x) + x((P+k)^2 - m^2 + i\epsilon)]^2}$$

$$= \frac{S^2}{2V^4} \int_0^1 dx \int_R d^4 k \frac{1}{[(k+xP)^2 - x^2 P^2 + xP^2 - m^2 + i\epsilon]}^2, \quad P^2 = S$$

$$l = k + xP$$

$$dl = dk$$

$$= \frac{S^2}{2V^4} \int_0^1 dx \int_R d^4 l \frac{1}{[l^2 - \Delta_x]^2}, \quad \Delta_x = x(x-1)S + m^2 - i\epsilon$$

Now we can Wick rotate contour

$$l^0 \rightarrow i l_E^0$$

$$l^2 \rightarrow -(l_E^0)^2 - \vec{l}^2$$

$$= \frac{i S^2}{2V^4} \int_0^1 dx \int_R d^4 l_E \frac{1}{[l_E^2 + \Delta_x]^2}$$

$$\int d^4 l_E = \int \frac{d^3 \vec{l}_E}{(2\pi)^3} \int dl_E^0$$

In the UV ( $l_E \rightarrow \infty$ ) we see this integral is logarithmically divergent  $\rightarrow \int dl_E \frac{l_E^3}{[l_E^2 + \Delta]^2} \sim \int_{\bullet}^{\wedge} \frac{dl_E}{l_E} \sim \ln \Lambda$

=> therefore, we know there must be a local operator which can absorb this divergence



Notice, this is a term w/ 4  $\theta$  fields, which did not exist

~~iA(s)~~ Let's use dimensional regularization to regulate this integral

$$iA(s) = \frac{iS^2}{2V^4} \int_0^1 dx \mu^{4-d} \int \frac{d^d l_E}{(2\pi)^d} \frac{1}{[l_E^2 + \Delta_x]^2} \quad d = 4 - 2\epsilon$$

↑  
we have to introduce dimensionful scale,  $\mu$ , to compensate for the change in dimensions of the integral  
~~The original  $\mathcal{L}$  in 4d~~

$$\mathcal{L} = \frac{1}{2} (2\pi\theta)^2 \frac{p^2}{V^2}$$

~~so the coefficient has dimension  $[\frac{1}{V^2}] = -2$   
but if we change the space time dimensions, the coupling must change dimensions, so that  $[\mathcal{L}] = d$   
so we can multiply~~

$$iA(s) = \frac{iS^2}{2V^4} \int_0^1 dx \mu^{2\epsilon} \cdot \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta_x}\right)^{2-\frac{d}{2}} \quad \begin{matrix} \frac{d}{2} = 2-\epsilon \\ 2-\frac{d}{2} = \epsilon \end{matrix}$$

$$A(s) = \frac{1}{2} \left(\frac{S}{4\pi V}\right)^2 \int_0^1 dx \mu^{2\epsilon} (4\pi)^\epsilon \cdot \left[\frac{1}{\epsilon} - \gamma_\epsilon + \mathcal{O}(\epsilon)\right]$$

We are interested in  $\lim_{\epsilon \rightarrow 0}$ , so drop all finite  $\epsilon$ -terms

$$(4\pi)^\epsilon = e^{\ln(4\pi)^\epsilon} = e^{\epsilon \ln 4\pi} \approx 1 + \epsilon \ln 4\pi$$

$$(4\pi)^\epsilon \left[\frac{1}{\epsilon} - \gamma\right] = \left[\frac{1}{\epsilon} - \gamma + \ln 4\pi + \mathcal{O}(\epsilon)\right]$$

$$A(s) = \frac{1}{2} \left( \frac{s}{4\pi V^2} \right)^2 \int_0^1 dx \left[ \frac{1}{\epsilon} - \gamma_E + \ln 4\pi + \ln \left( \frac{\mu^2}{\Delta_x} \right) \right]$$

$$= \frac{1}{2} \left( \frac{s}{4\pi V^2} \right)^2 \left[ \frac{1}{\epsilon} - \gamma_E + \ln 4\pi - \ln \left( \frac{s}{\mu^2} \right) - \int_0^1 dx \ln \left( x^2 - x + \frac{m^2 - i\epsilon}{s} \right) \right]$$

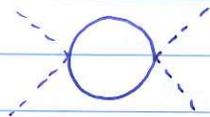
$$\int_0^1 dx \ln \left( x^2 - x + \frac{m^2 - i\epsilon}{s} \right) = \begin{cases} -2 + \ln \frac{m^2}{s} + \sqrt{\frac{4m^2}{s} - 1} \left( 2 \tan^{-1} \left( \sqrt{\frac{4m^2}{s} - 1} \right) - \pi \right), & s < 4m^2 \\ -2 + \ln \frac{m^2}{s} + \sqrt{1 - \frac{4m^2}{s}} \ln \left( \frac{1 + \sqrt{1 - \frac{4m^2}{s}}}{1 - \sqrt{1 - \frac{4m^2}{s}}} \right) - i\pi \sqrt{1 - \frac{4m^2}{s}}, & s > 4m^2 \end{cases}$$

$$A(s) = \frac{1}{2} \left( \frac{s}{4\pi V^2} \right)^2 \left[ \frac{1}{\epsilon} - \gamma_E + \ln 4\pi - \ln \frac{m^2}{\mu^2} - 2 + \begin{cases} i\sigma \left( 2 \tan^{-1} (i\sigma) - \pi \right) & s < 4m^2 \\ \sigma \ln \left( \frac{1 + \sigma}{1 - \sigma} \right) - i\pi \sigma & s > 4m^2 \end{cases} \right]$$

$$\sigma = \sqrt{1 - \frac{4m^2}{s}}$$

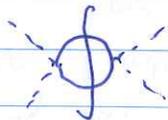
$s < 4m^2$   $A(s)$  is purely real

$s > 4m^2$   $A(s)$  develops imaginary component



for  $s > 4m^2$ , the "internal"  $p$  states have enough energy to go "on-shell" (become real and propagate asymptotically far)

This is related to the optical theorem which relates  $\text{Im} A \propto |A|^2$



"denotes "cut" = imaginary contribution

What else can we say about this amplitude?

- the amplitude is dimensionless,

$$\left(\frac{S}{V^2}\right)^2$$

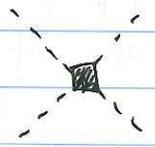
as required for  $2 \rightarrow 2$  scattering in 4d

- the amplitude has a divergence which must be renormalized (dim. reg regulated it)

$$A(s) \propto \left(\frac{S}{V^2}\right)^2 \cdot \frac{1}{\epsilon}$$

we know the operator must have 4  $\theta$  fields (new operator), and it is proportional to  $s^2$   
 $\Rightarrow$  4 derivatives

$$\delta \mathcal{L}_5 \propto \frac{C_4}{V^4} (\partial_\mu \theta)^2 (\partial_\nu \theta)^2$$



with this new operator, we can now Renormalize the theory (regulate = make finite, renormalize  $\Rightarrow$  make  $\mu$  independent)

$C_4$  will contain both infinite  $\left(\frac{1}{\epsilon}\right)$  terms as well as finite terms. It must also depend on  $\mu$ , in such a way

$$\frac{\partial}{\partial \mu} (A(s) + \delta A(s)) = 0$$

as  $s \rightarrow 0$ , the amplitude

Why would we go through all this trouble (polar form)?

$$\phi = \frac{\rho + i\theta}{\sqrt{2}} e^{i\theta/v}$$

\* The interactions of  $\theta$  (Goldstone bosons) are now manifestly momentum dependent

as  $s \rightarrow 0$ , these modes decouple from the theory.

This is a crucial observation

Goldstone Bosons can only be derivatively coupled to other fields

\* This was a good excuse to introduce an "Effective Field Theory" (EFT)

- this theory provides a good description of the dynamics for

$$s \ll 4\pi V^2$$

so we see the quantity which sets the scale of SSB, also dictates the range of validity of our EFT

- naturally, a dimensionless ratio of energy scales emerges

$$E = \frac{s}{4\pi V^2}, \text{ which provides a perturbative}$$

expansion parameter

We see the new operator

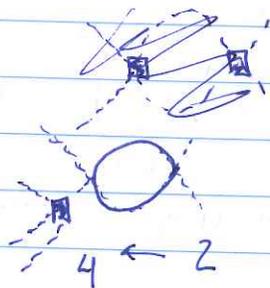
$$\delta \mathcal{L} \propto \frac{c_4(\mu)}{v^4} (\partial_\mu \theta \partial_\nu \theta)^2$$

(why  $4\pi v^2$  instead of  $v^2$ ?) loop integrals will always produce appropriate factors of  $4\pi$  s.t. the  $\epsilon = \frac{S}{4\pi v^2}$

$$\delta \mathcal{L} = \frac{c_4(\mu)}{(4\pi v^2)^2} (\partial_\mu \theta \partial_\nu \theta)^2$$

so this operator is suppressed compared to the the leading  $\mathcal{L}$  by a factor of  $\epsilon$

If we were to use this operator in a loop



the corrections would need ~~to be~~ even new operators of higher dimension, but further suppressed in  $\epsilon$

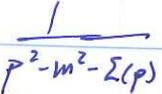
So even though this EFT is non-renormalizable, w/ an infinite tower of operators, the relevance of these operators is dictated by an expansion in  $\epsilon = \frac{S}{4\pi v^2}$

Just for fun, lets compute the self energy correction to  $\theta$

$$\mathcal{L} > \frac{1}{2} (\partial_\mu \theta)^2 \frac{p^2}{v^2}$$


 $= -i \delta \Sigma_\theta(p) = -\frac{i}{2v^2} (-ip)(ip) \int_R d^4k \frac{i}{k^2 - m^2}$   
 $= (-i)^2 (i)^4 \frac{p^2}{2v^2} \int_R d^4k \frac{1}{k^2 - m^2}$   
 but  $p^2 = 0!$

sign comes from self-energy sum of 1PI diagrams


 $\frac{1}{p^2 - m^2 - \Sigma(p)}$

Just as we expect, since the  $\theta$  is a goldstone boson, it's mass is protected by a symmetry to be exactly 0.

What about the self-energy correction to  $\rho$


 $= -i \delta \Sigma_\rho(p) = -\frac{i}{2v^2} \int_R d^4k \frac{(-ik)(+ik) i}{k^2 - m_\theta^2 + i\epsilon}$   
 $= \frac{1}{2v^2} \int_R d^4k \frac{k^2}{k^2}$   
 $= \frac{1}{2v^2} \int_R d^4k \ 1$   
 $= 0! \text{ (dim-reg magic)}$

There is no scale for the integral to depend on, therefore, in dim-reg this integral is 0, even though it is superficially  $\Lambda^4$  divergent